2024
SRCMUJ
2nd Semester Examination
M. Sc.
Mathematics
MTM-202
Functional Analysis

Full Marks: 40 **Time: 2 hours**

The figures in the margin indicate full marks. Candidates are required to give their answers as far as practicable. Notations have their usual meaning.

1. Answer any four of the following questions: **4×2**

(a) State Hahn-Banach theorem.

(b) Show that every normed space can be embedded as a dense subspace of a Banach space.

(c) Let H be a Hilbert space and $\{\varphi_i\}_{i=1}^{\infty}$ be an orthonormal system in H. Show that $\|\varphi_n - \varphi_m\| = \sqrt{2}$ for $m \neq n$.

(d) Let H_1, H_2 be two Hilbert spaces and $S : H_1 \to H_2;\ T : H_1 \to H_2$ be two bounded linear operators. Then show that $(ST)^* = T^* S^*$.

(e) Show that if the sequence $\{x_n\}$ in a normed space X is weakly convergent to $x_0 \in X$, then $\liminf\limits_{n\to\infty} \|x_n\| \geq \|x_0\|$.

(f) Let $T \in BL(H)$ and $T \geq 0$ where H is a Hilbert space. Show that $\|Tx\|^2 \leq \|T\| \leq \langle Tx, x\rangle$ for all H.

2. Answer any four of the following questions: **4×8**

(a) (i) Consider the Hilbert space $H = L^2[-1, 1]$ equipped with the usual scalar product:

$$\langle f, g\rangle = \int_{-1}^{1} \overline{f(t)}\, g(t)\, dt, \quad f, g \in H.$$

Let $E = \{x \in H : f(-t) = f(t),\ t \in [-1, 1]\}$. Then

(I) Show that E is closed in H. Find $E^{\perp}$.

(II) Calculate the distance from h to E for $h(t) = e^t$. 5

(ii) Show that the operator $P \equiv -i\dfrac{d}{dx}$ is self-adjoint on $L_2(\mathrm{IR})$. 3

(b) (i) State and prove Banach-Steinhuss theorem. 1+ 5

(ii) Give an example to show that $C[0, 1]$ equipped with the L^1 - norm

$$\|f\|_1 = \int_0^1 |f(x)|\, dx, \quad f \in C[0, 1]$$

is not a Banach space. 2

(c) Let a, b be real numbers such that $a < b$. Consider the Hilbert space $L^2[a, b]$ over IR and the operator $T : L^2[a, b] \to \mathrm{IR}$ be defined by $T f = \int_a^b f(x)dx, \quad f \in L^2[a, b]$.

(i) Show that T is bounded. Compute $\|T\|$.

(ii) According to the Riesz's theorem, there exists a function $g \in L^2[a, b]$ such that $T f = \langle f, g \rangle$ for all $f \in L^2[a, b]$. Find such a function g and verify that $\|g\|_{L^2} = \|T\|$. 4 + 4

(d) (i) Prove that the set of all bounded linear operators, $B(X, Y)$, is a Banach space if Y is a Banach space. 4

(ii) Show that in any finite dimensional vector space, strong convergence and weak convergence are equivalent. 4

(e) (i) Show that every infinite dimensional separable Hilbert space H is isometrically isoporphic to the sequential space l_2. 6

(ii) Give an example to show that every closed and bounded linear space may not be compact. 2

(f) (i) Let $S \in BL(H)$, where H is a Hilbert space. Prove that for all $x, y \in H$, 4

$$< Sx, y >= \frac{1}{4}\sum_{n=0}^{3} i^n < S(x + i^n y), (x + i^n y) > .$$

(ii) State and prove Riesz-Fischer Theorem. 4