2024
SRMUJ
M. Sc.
1st Semester Examination
Mathematics
MTM-101
Real Analysis-I

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks. Candidates are required to give their answers as far as practicable. Notations used here have their usual meaning.

1. Answer any four of the following questions: **2 × 4**

(a) Define equicontinuity with an example.

(b) Give an example of a function f which is not continuous on a closed interval but f is a function of bounded variation on that interval.

(c) Let X be a measurable space and $\chi_E : X \to \mathbb{R}$ be a measurable function, where

$$\chi_E(x) = \begin{cases} 1 \; if \; x \in E \\ 0 \; if \; x \notin E \end{cases}.$$

Is E a measurable set in X?

(d) Show that $\{f_n\}$, where $f_n(x) = \frac{\log(1+n^3x^2)}{n^2}$ is uniformly convergent on $[0, 1]$.

(e) Evaluate $\int_1^4 (x - [x])\, dx^2$.

(f) Define Borel set with example.

2. Answer any four of the following questions: **8 × 4**

(a) (i) Let $f : [a, b] \to \mathbb{R}$ be a function of bonded variation on $[a, b]$ and let V be its corresponding variation function. Then show that f is continuous at a point $c \in [a, b]$ if and only if V is continuous at c.

(ii) Check whether the function $f(x) = |5x - 7| + |x|$ on $[0, 3]$ is a function of bounded variation or not. If so, also find the variation function of f on $[0, 3]$. 4 + 4

(b) (i) Let $\{f_n\}$ be a sequence of equicontinuous, real valued, uniformly bounded continuous functions on $\mathbb{R}$. Show that $\{f_n\}$ has a convergent subsequence which converges uniformly on any bounded subset of $\mathbb{R}$.

(ii) Show that Cantor set is an uncountable set. 4 + 4

(c) (i) If f is measurable, then show that for every extended real number α, the set $E(f = \alpha) = \{x \in E : f(x) = \alpha\}$ is measurable.

(ii) Let m be a measure on a σ – algebra of subsets of X . Show that the outer measure μ^* induced by μ is countably subadditive. 4 + 4

(d) (i) Let $f_n: X \to \mathbb{R}^*$ be measurable for $n = 1, 2, 3, \ldots$ Then show that $\liminf_{n\to\infty} f_n$ is a measurable function on X.

(ii) Construct a non-measurable subset of IR. 4 + 4

(e) (i) Let ϕ be a continuous and strictly monotone function on $[\alpha, \beta]$. If f is a continuous function on $[a, b]$, then show that $\int_a^b f(x)\,dx = \int_\alpha^\beta f(\phi(y))\,d(\phi(y))$, where $a = \phi(\alpha),\ b = \phi(\beta)$.

(ii) Let $f: [0, 1] \to \mathbb{R}$ be defined by $f(0) = 0$, $f(1) = 1$ and

$f(x) = 0$, when x is rational

$= \frac{1}{q^3}$, when $x = \frac{p}{q}$ where $p,\ q$ are integers prime to each other and $q > p$.

Prove that f is a function of bounded variation on $[0, 1]$. 4 + 4

(f) (i) Prove that if $f:[a,b]\to \text{IR}$ is continuous, then f is measurable. Is the converse true? Justify your answer.

(ii) Suppose K is a compact subset of $\mathbb{R}$ and the sequence of functions $f_n(x)$ is continuous on K. If each f_n is pointwise bounded and equicontinuous on K, then show that f_n is uniformly bounded on K. 5 + 3

P.Y.Q 2024