2024
SRMUJ
M. Sc.
1st Semester Examination
Mathematics
MTM-103
Ordinary Differential Equations and Special Functions

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks. Candidates are required to give their answers as far as practicable. Notations used here have their usual meaning.

1. Answer any four of the following questions: **2×4**

(a) Explain the concept of Green's function for the non-homogeneous equation $Lu[x] = f(x)$, where L is Sturm-Liouville operator, subject to some boundary conditions at the end points of the interval $a \le x \le b$.

(b) Find the general solution of the system of equation: $\dot{x} = -3x + 4y$; $\dot{y} = -2x + 3y$.

(c) Let $P_n(z)$ is the Legendre polynomials of degree n, then expand $f(z) = z^2 + 1$ in terms of $\sum c_r\ P_r(z)$.

(d) Explain, with an example, the Fuchsian type differential equation.

(e) Examine that whether infinity is a regular singular point for Bessel's differential equation or not.

(f) When a boundary problem is a Sturm-Liouville problem.

2. Answer any four of the following questions: **8×4**

(a) Determine a fundamental matrix for the linear vector differential equation

$\frac{d\vec{x}}{dt} = \begin{pmatrix} 5 & 2 & -2 \\ 7 & 0 & -2 \\ 11 & 1 & -3 \end{pmatrix} \vec{x},\ \vec{x} = (x_1 \quad x_2 \quad x_3)^T$ and find its general solution. Also determine the unique solution $\vec{\phi}$ that satisfies the initial condition $\vec{\phi}(\vec{0}) = (1 \quad 0 \quad 4)^T$. $5 + 1 + 2$

(b) (i) Find the series solution of the differential equation $z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - 1)w = 0$ in powers of z.

(ii) Show that for the Legendre polynomials of degree n, $P_n(z)$, $P_n(-z) = (-1)^n P_n(z)$

6 + 2

(c) (i) Construct the Green's function for the boundary value problem

$$y''(x) - \frac{1}{x} y'(x) = 1, \text{ with } y(0) = y(1) = 0.$$

And hence solve the equation.

(ii) Write down the conditions for the existence and uniqueness of solutions to a system of n linear 1st order ordinary differential equations? 6 + 2

(d) (i) Find the characteristic values and the characteristic functions of the Sturm-Liouville system $\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x} y = 0$; $y'(1) = 0$, $y'(e^{2\pi}) = 0$ where we assume that λ is a non-negative parameter.

(ii) For the Bessel function of the first kind of order n, $J_n(z)$, show that $\frac{d}{dz}\left(z^{-n} J_n\right) = -z^{-n} J_{n+1}$. 5 + 3

(e) (i) If α and β are the roots of the equation $J_n(z)=0$ then show that

$$\int_0^1 z J_n(\alpha z) J_n(\beta z) dz = \begin{cases} 0, & if\ \alpha \neq \beta \\ \frac{1}{2}[J_n'(\beta)]^2 & if\ \alpha = \beta \end{cases}$$

(ii) If $z > 1$, then prove that $P_n(z) < P_{n+1}(z)$. 6 + 2

(f) (i) Find the general solution of the non-homogeneous system,

$$\frac{dX}{dt} = \begin{pmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{pmatrix} X + \begin{pmatrix} -5t-6 \\ -4t+23 \\ 2 \end{pmatrix} \text{ where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(ii) Prove that, $P_n(z) = P_{-n-1}(z)$. 6 + 2

P.Q. 2024