2024
SRMUJ
M.Sc.
1st Semester Examination
Mathematics
MTM-104
Abstract Algebra

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable. Notations have their usual meaning.

1. Answer any four of the following questions: **4 × 2**

(a) Find all abelian group of order 81.

(b) Is the group S_4 solvable? Justify your answer.

(c) If F is a finite field of characteristic p and $\left(F: \mathbb{Z}_p\right) = n$. Show that F contains p^n elements.

(d) Is a primary ideal always a prime ideal? Justify your answer.

(e) Prove that every finite field is perfect.

(f) Define degree of field extension. Find $(\mathbb{Q}(\sqrt{3}, \sqrt{5}): \mathbb{Q})$.

2. Answer any four of the following questions: **8 × 4**

(a) (i) Prove that every finite p-group is nilpotent.
(ii) Define separable extension with example. Let K be a field of characteristic p (> 0). Show that an irreducible polynomial $f(x) \in K[x]$ is inseparable if and only if $f(x) = g(x^p)$ for some $g(x^p) \in K[x^p]$. 4 + 4

(b) (i) State and prove fundamental theorem of field extension.

(ii) Find the splitting field S of $x^2 + x + [1]$ over $\mathbb{Z}_5$. Find $(S: \mathbb{Z}_5)$ and a basis for $S/\mathbb{Z}_5$. 6 + 2

(c) (i) Show that every subgroup of a solvable group is solvable.
(ii) Let F/K be a field extension and $\alpha \in F$ be algebraic over K. Let $f(x)$ be the minimal polynomial of α over K. Then $f(x)$ is the monic polynomial of smallest degree in $K[x]$ having α as a root.
(iii) Construct a Galois field with 9 elements. 4 + 2 + 2

(d) (i) State and prove first Sylow theorem.

(ii) Let G be a finite group and H be a subgroup of G of index n such that $|G|$ does not divide $n!$. Show that G contains a non-trivial normal subgroup. 5 + 3

(e) (i) Let K be a field and n be a positive integer such that $chK \nmid n$. Let G be the set of all n^{th} roots of unity in K. Then show that G is a multiplicative cyclic group and $|G|$ divides n. Also, show that if $x^n - 1$ splits into linear factor in $K[x]$ then $|G| = n$.

(ii) Define Galois extension. Find the Galois group of the polynomial $x^4 - 5 \in \mathbb{Q}[x]$. 4 + 4

(f) (i) Let G be a group of all $n \times n$ real matrices which are invertible where $n \geq 3$. Show that G is not solvable.

(ii) In the field extension $\mathbb{R}/\mathbb{Q}$, show that π^2 is transcendental over $\mathbb{Q}$.

(iii) Let F/L and L/K be two field extensions such that $(F:L) = m$ and $(L:K) = n$. Prove that $(F:K) = mn$. 2 + 2 + 4