**2024**
**SRMUJ**
**3rd Semester Examination**
**M.Sc.**
**Mathematics**
**MTM-304**

**Full Marks: 40** **Time: 2 Hours**

The figures in the margin indicate full marks. Candidates are required to give their answers as far as practicable. Notations used here have their usual meaning. Use **separate sheets** for each unit.

## Unit – I [Linear Algebra]

**1. Answer any two of the following questions:** **2 × 2**

(a) Define normal operator. Let $V$ be an inner product space and $T$ be a normal operator on $V$. Show that $(T - cI)$ is also a normal operator for all $c \in F$ (scalar field).

(b) Suppose A ≠ I be a square matrix for which $A^3 = I$. Determine whether A is not similar to a diagonal matrix if A is a matrix over (i) the real field $\mathbb{R}$ and (ii) the complex field $\mathbb{C}$.

(c) Prove that a bilinear form $f(X, Y) = X^t AY$ is symmetric if and only if $A$ is symmetric.

(d) Write down the differences between Jordan canonical form and rational canonical form of a matrix.

**2. Answer any two of the following questions:** **2 × 8**

(a) (i) Let $V$ and $W$ be finite dimensional vector spaces over the field $F$ with ordered bases $\beta$ and $\gamma$ respectively. For any linear transformation $T: V \to W$, prove that the mapping $T^t: W^* \to V^*$ defined by $T^t(g) = gT$ for all $g \in W^*$ is a linear transformation with the property that $[T^t]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$.

(ii) Define dual space and dual basis of a vector space $V$. Find the dual basis $\emptyset_1, \emptyset_2, \emptyset_3$ of the basis $v_1 = (1, -1, 3)$, $v_2 = (0, 1, -1)$, $v_3 = (0, 3, -2)$ of $\mathbb{R}^3$. 3 + 5

(b) (i) State and prove the Spectral theorem. 5

(ii) Define a bilinear form $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $f(\alpha, \beta) = 2a_1b_1 + 3a_1b_2 + 4a_2b_1 - a_2b_2$, where $\alpha = (a_1, b_1), \beta = (a_2, b_2) \in \mathbb{R}^2$. Find the matrix representation of $f$ with respect to the standard ordered bases. Also find $f$ with respect to the ordered bases $(1, 1)$ and $(1, -1)$. 3

(c) (i) Let V be a vector space of dimension 8 over the rational field $\mathbb{Q}$ and let T: V → V be a linear operator with minimal polynomial $m(t) = (t^4 - 4t^3 + 6t^2 - 4t - 7)(t - 3)^2$. Find all possible rational canonical form M of T.

(ii) Prove that the minimal polynomial of a matrix always divides its characteristic polynomial.

(iii) Discuss the nature of eigen values of a linear operator on a vector space $V$ in the following cases when (A) $T$ is unitary, (B) $T$ is self-adjoint, (C) $T$ is skew-adjoint and (D) $T$ is positive definite. 2 + 2 + 4

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(d) (i) What is nilpotent operator? Give an example.

(ii) Let $V$ be a vector space of polynomial functions in two real variables $x$ and $y$ of degree 2 and $\alpha = \{1, x, y, x^2, y^2, xy\}$ is an ordered basis for $V$. If $T$ be a linear operator on $V$ defined by $T\big(f(x,y)\big) = \frac{\partial}{\partial x} f(x,y)$, where $f(x,y) = x + 2x^2 - 3xy + y$. Find the Jordan canonical form of $T$. Also find a Jordan canonical basis for $T$. 2 + 6

## Unit – II [Manifold Theory]

**3. Answer any two of the following questions:** **2 × 2**

(a) Show that the solution set $S$ of $x^3 + y^3 + z^3 = 1$ in $\mathbb{R}^3$ is a manifold of dimension 2.

(b) If $f$ is a smooth function on a smooth manifold M, then show that $d^2 f = 0$ , where '$d$' denotes the exterior derivative.

(c) Let $X = (2, 3, 0)$ be a vector in $\mathbb{R}^3$. Find $X_p(f)$ for $P = (-2, \pi, 1)$, where $f = x^1 x^3 \cos x^2$.

(d) Are circle and square diffeomorphic to each other? Justify your answer.

**4. Answer any two of the following questions:** **2 × 8**

(a) (i) Define chart on a manifold $M$. When two chart are compatible?

(ii) Find the functional relation between two local co-ordinate systems defined in the overlapping region of a topological manifold.

(iii) State regular set level theorem. Examine whether the subset $S$ of $\mathbb{R}^3$ defined by the two equations $x^3 + y^3 + z^3 = 1$, $x + y + z = 0$ is a regular level set or not. 2 + 2 + 4

(b) (i) What is integral curve? Consider the vector field $X_{(x,y)} =< y, -x >$ on $\mathbb{R}^2$. Find the integral curve $c(t)$ of $X$ starting at the point $(1, 0) \in \mathbb{R}^2$.

(ii) Show that $\varphi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\varphi(t,\ p) = (p^1 + t,\ p^2 + t,\ p^3 + t)$ is a one-parameter group of transformations on $\mathbb{R}^3$. Find its generator. 5 + 3

(c) (i) If $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is a smooth manifold with boundary then find Int A and $\partial A$. Also write their dimensions as a smooth manifold.

(ii) Show that the *n*-dimensional real projective space $\mathbb{R}P^n$ is a smooth manifold of dimension *n*. 2 + 6

(d) (i) Obtain a necessary and sufficient condition so that the 1-forms $\omega_1, \omega_2, \cdots, \omega_r$ are linearly dependent.

(ii) Let $M$ be a smooth *n*-manifold and $f \in C^\infty(M)$. Prove that $d(f\omega) = df \wedge \omega + f d\omega$ for any $k$-form $\omega$ on $M$. 5 + 3

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